

Consensus Analysis of Wireless Multi-Agent Systems Over Fading Channels

Salil Kumar Malla

Department of Electronics and Telecommunication, College of Engineering Bhubaneswar

Abstract— The likelihood of consensus in wireless multi-agent systems (WMASs) is presented in this letter. We derive the probability expression that a WMAS with N agents obtains consensus by starting with the connection probability of random graphs and accounting for link outages between agents. We verify the exact expression produced by simulation outcomes. Additionally, we demonstrate that the appropriate diversity order is $N - 1$, meaning that even with the increased communication requirements between agents, a larger agent count will increase the likelihood of consensus. Lastly, we note that the number of agents in a WMAS and the signal-to-noise ratio (SNR) level have a significant impact on the consensus speed.

I. INTRODUCTION

SYSTEMS which contain multiple agents that can communicate with each other over wireless channels are referred to as wireless multi-agent systems (WMASs). An agent contains independent and computer-based dynamic systems. Air traffic control and unmanned autonomous space vehicles are some examples of these systems. In general, WMASs aim at accomplishing a task as a group in the shortest possible time. Thus, it is important for these systems to use the information gathered from every agent and to reach a common decision.

Agents can store real-time information about some variables. Depending on the consensus protocol, this information will be updated until a consensus is reached. Consensus problems have been first discussed in 1974 by DeGroot in [1]. In the recent years, this problem was studied in the MAS context by Olfati-Saber *et al.*, in [2] and [3]. In [4] and [5], the

consensus problem and event-triggered communication strategy in MASs are analyzed, and in [4] the proposed method is applied to a vehicle platoon control.

WMASs transfer the information symbols over the wireless channels between agents. Due to obstacles and/or other environmental factors, signals can be reflected, refracted, and scattered. These effects are known to cause multipath fading, which is known to affect the communication channels. Due to the presence of multipath fading, temporary disconnections of wireless links can be encountered frequently and randomly. Such random disconnections may cause faulty decisions in the consensus process. A WMAS model with four agents (drones) and the network topology of this WMAS are shown in Fig. 1. The dotted lines represent the communication links between agents.

Convergence of consensus algorithms and the consensus performance of WMASs over fading channels are studied in [6] and [7], respectively. In [8], consensus problems of multi-agent systems for superposition property of the wireless channel are studied. The impact of jamming attacks in WMASs is studied in [9]. However, the connection between consensus problems of WMASs in wireless channels and random graphs has not yet been investigated. Random graphs and the probability of having a connected graph in random graphs were first discussed in 1959 by Gilbert in [10]. Based on these findings, our work introduces the probability of reaching consensus in WMASs for the first time in the literature, shedding some light on the expected behaviors in the presence of wireless channels.

In this letter, by using random graphs, the effect of random disconnections on the consensus protocol will be modeled and examined. Starting with a link outage probability, we derive the consensus probability of N agent WMASs under symmetric topologies, where the average signal-to-noise ratio (SNR) between the agents are the same. We verify the resulting expressions through Monte Carlo simulations. Further, we show that the diversity order of the consensus outage probability expression is $N - 1$, indicating that despite the additional

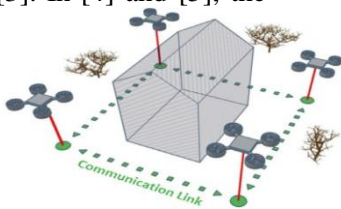


Fig. 1. A WMAS model with four agents.

communication complexity, increasing the number of agents contributes significantly to the consensus performance. Finally, we compare the duration of consensus processes.

This letter is organized as follows. In Section II, the relation between the link outage and the wireless channel is explained. Section III introduces the consensus algorithm and the network model. In Section IV, the consensus outage probability and the corresponding diversity order are studied. In Section V, numerical results are presented. Finally, in Section VI, our conclusions and future work recommendations are given.

II. THE NETWORK MODEL

The impact of the wireless channels cannot be neglected while analyzing the performance of a WMAS system. Transmitted signals can be affected by both the fading channel and the ambient interference [11]. The effect of the wireless impairment can be taken into account in terms of the link outage probability between the i^{th} agent and the j^{th} agent as

$$P_{out}(\gamma_{ij}, R) = \Pr R < \log_2 1 + \gamma_{ij} \quad , \quad (1)$$

where $\Pr(\cdot)$ denotes the probability function, R represents the target data rate, and γ_{ij} represents the instantaneous SNR.

The instantaneous SNR is related to the average SNR ($\bar{\gamma}$), and the instantaneous channel tap (h_{ij}) as $\gamma_{ij} = |h_{ij}|^2 \bar{\gamma}_{ij}$. So, when

$$|h_{ij}|^2 \geq 2^R - 1 \bar{\gamma}_{ij}^{-1} \quad (2)$$

holds, then reliable communication is possible between the i^{th} and the j^{th} agents. Otherwise, a reliable communication

cannot be guaranteed.

The fading channel tap can be modeled as a random variable depending on the environmental characteristics. The pres-

ence or the absence of a line-of-sight (LoS) component can be reflected in the corresponding probability density function (PDF). In the numerical analysis below, we consider the

Rayleigh model which is commonly used in the literature. This model assumes the channel to behave like a heavily built-up urban environment with no LoS component. The link outage probability in presence of a Rayleigh fading channel is

$$P_{out}(\bar{\gamma}_{ij}, R) = 1 - \exp \left(- \frac{2^R - 1}{\bar{\gamma}_{ij}} \right) \quad . \quad (3)$$

III. NETWORK TOPOLOGY AND CONSENSUS

A wireless network topology and the linear consensus protocol can be represented by using the graph theory, where a graph model such as $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ can be used. Here, $\mathcal{V} = \{v_1, \dots, v_N\}$ denotes vertices, representing N agents in the WMAS system, while $i \neq j$ all connections between agents (v_i, v_j) are contained in $\mathcal{E} \in \{\dots, e_{ij}, \dots\}$, the edge set. e_{ij} denotes the edge between v_i and v_j . The set of neighbors of v_i can be expressed as $N_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$. For a fully connected graph, neighbor set of v_i will be $N_i = \{1, \dots, N\} \setminus \{i\}$.

Otherwise, $a_{ij} = 0$ [12]. In case of a weightless graph, A can only have binary values $\{0, 1\}$.

In this work, communication channels are assumed to be bidirectional, i.e., $a_{ij} = a_{ji}$. This channel reciprocity assumption, which is shown to be valid in the current time division duplexing (TDD) systems, implies that if the i^{th} agent can receive information from the j^{th} agent, then it can also transmit its information to the j^{th} agent. In a graph with N nodes, there are $N(N-1)/2$ possible edges (bidirectional channels) and 2^N possible graphs that can be generated [10].

In addition to the adjacency matrix, a degree matrix D can be defined. This matrix is a diagonal matrix whose elements are $[D]_{ii} = d_{ii} = \sum_{j \in N_i} a_{ij}$ [12]. Here, the degree is the number of receiving channels a node has. Since bidirectional communication channels are used in this work, the matrix D also shows the number of neighbors each agent has. Based on A and D , the Laplacian matrix for G is defined as [3]

$$L(G) = D - A. \quad (4)$$

In WMASs, linear time average consensus protocol, $\chi(x) = (1/N) \sum_{i=0}^{N-1} x_i(t_0)$, is commonly used. In this protocol,

agents perform the averaging over the received values from their neighbors and agents expect the consensus value to reach $\chi(x)$. Consensus is reached when the states of each agent

reaches a common value α . To reach consensus, each state's value should constantly be updated by the state feedback. In zero communication delay cases, consensus protocol

for the i^{th} agent can be written as follows [3]

$$\dot{x}_i(t) = u_i(t). \quad (5)$$

Here $u_i(t)$ is defined as the state feedback and its value can be found by equation

$$u_i(t) = \sum_{v_j \in N_i} a_{ij} (x_i - x_j). \quad (6)$$

This equation can be expressed in matrix notation as

$$\dot{x}(t) = -L(G)x(t). \quad (7)$$

The protocol stated in (5) can be modelled for discrete time cases by Markov chains as follows

$$x(k+1) = P_{\epsilon} x(k), \quad (8)$$

where P_{ϵ} stands for a Perron matrix [2]. This matrix is non-

The link status between two agents depends on the communication channel, specifically, the instantaneous SNR, γ_{ij} . To show whether the links in the corresponding edges are connected, an adjacency matrix A can be defined. Elements of this matrix can be shown as $[A]_{ij} = a_{ij}$. If there is a successful communication between the i^{th} and the j^{th} agent (i.e., the corresponding link is in non-outage status), then $a_{ij} = 1$.

negative, row-stochastic and can be defined as

$$P_{\epsilon} = I - \epsilon L(G), \quad (9)$$

where ϵ is defined as step size. If the graph is strongly connected, P_{ϵ} is irreducible and primitive. A strongly connected graph is a directed graph that includes a path, where every node is connected to every other node in the graph.

During the consensus process, agents transmit their information respectively. When the last agent transmits its information, all agents in the network update their information as explained in (8) and (9). This operation corresponds to one iteration in a WMAS. This process goes for n iterations and is terminated when the information in every agent reaches the value of α . Termination of the process is determined via the state feedback. As the iterations progress the system converges toward consensus if $u_i(t)$ approaches 0.

There are several conditions for systems that are represented by graphs to reach consensus. These conditions are:

- 1) As given in [13, Th. 3.2], a WMAS reaches consensus if there exists a rooted spanning tree in the graph $G = (V, E, A)$. A rooted spanning tree is a tree that contains all nodes in a graph with a fixed root.
- 2) In [2], Theorem 1 states that if the graph $G = (V, E, A)$ is strongly connected, $\text{rank}(L(G)) = N - 1$ is satisfied. A detailed proof for this condition can be found in [2, pp. 1530–1531] and in [3, p. 221].
- 3) Reference [2, Th. 5] states that a symmetric and strongly connected WMAS (G, A) reaches average consensus if and only if $\mathbf{1}^T L(G) = 0$ is satisfied. Vector $\mathbf{1}$ is defined as $\mathbf{1} = (1, \dots, 1)^T$.

IV. CONSENSUS PROBABILITY AND DIVERSITY ORDER

Consensus outage probability is the probability that consensus probability falls below the minimum acceptable consensus performance. In this section, we study the consensus outage probability and the associated diversity order of WMASs with N agents under symmetric topologies, where $\bar{V}_{i,j} = \bar{V} \forall i, j$.

Theorem 1: For symmetric and strongly connected graphs

of an N -agent WMAS, the consensus outage probability can be obtained as

$$P_{Co}(N, \bar{V}) = 1 - \sum_{r_1, \dots, r_N} \frac{(-1)^{n-1} (n-1)! N! \times P_{out}(\bar{V}, R)^{(N^2 - 1^2 r_1 - \dots - N^2 r_N)/2}}{r_1! \dots r_N! (1!)^{r_1} \dots (N!)^{r_N}}, \quad (10)$$

where r_1, r_2, \dots, r_N are non-negative integers, and $N = r_1 + 2r_2 + \dots + Nr_N$ and $n = r_1 + r_2 + \dots + r_N$.

Proof: In random graphs, the process of choosing the edge connections is random. Every decision is independent of each other and may create different graph structures. The common probability of joining is denoted as $p(\bar{V})$ and erasure probability is denoted as $q(\bar{V})$ where $q(\bar{V}) = 1 - p(\bar{V})$. The probability of a connected graph and the corresponding probability of reaching a consensus can be obtained as [10]

$$P(N) = \sum_{C_{N,L}} C_{N,L} p(\bar{V})^L q(\bar{V})^{-L+N(N-1)/2}. \quad (11)$$

$C_{N,L}$ denotes the number of connected graphs that have N nodes and L edges. In [10], the generating series for $C_{N,L}$ is expressed as

$$\sum_{N,L} C_{N,L} \phi^N y^L = \log \left(1 + \sum_{i=1}^{\infty} \frac{\phi^i (1+y)^{i(i-1)/2}}{i!} \right) \quad (12)$$

Here ϕ is an auxiliary variable and $y = \frac{p(\bar{V})}{q(\bar{V})}$. By using (11)

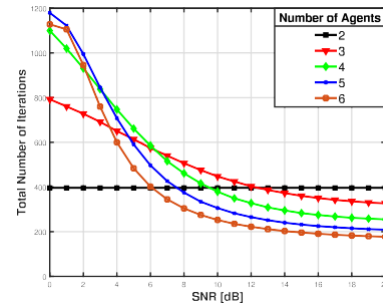


Fig. 2. The total number of iterations for different SNR levels.

In general, the diversity order (gain) indicates the slope of the average error probability versus the average SNR curve [14]. To observe the effect of wireless channels, a diversity order analysis is made. In this context, we investigate the diversity order of the consensus outage probability of the WMASs. Following the same approach in [15], we define the diversity order of an N agent WMAS d as follows

$$\lim_{\bar{y} \rightarrow \infty} \frac{\log P_{Co}(N, \bar{V})}{\log(\bar{y})} = -d. \quad (14)$$

Corollary 1: The diversity order of an N agent WMAS is directly related with the rank of the Laplacian matrix $L(G)$,

and can be obtained as $d = N - 1$.

Proof: Following the steps in [10], the consensus outage probability becomes

$$P_{Co}(N, \bar{V}) = Nq(\bar{V})^{N-1} - O(N^2 q(\bar{V})^{3N/2}), \quad (15)$$

where the $O(\cdot)$ notation in the second term of the summation represents the collection of terms with order higher than $3N/2$. Restating (14), we obtain

$$d = - \lim_{\bar{y} \rightarrow \infty} \frac{\log(Nq(\bar{V})^{N-1} - O(N^2 q(\bar{V})^{3N/2}))}{\log(\bar{y})}. \quad (16)$$

As $\bar{y} \rightarrow \infty$, $q(\bar{V}) \rightarrow 0$. Since $\bar{y} \rightarrow \infty$, $q(\bar{V}) \rightarrow 0$ and $N > 0$, $Nq(\bar{V})^{N-1} > N^2 q(\bar{V})^{3N/2}$. Thus, (16) becomes

$$\lim_{\bar{y} \rightarrow \infty} \frac{\log(Nq(\bar{V})^{N-1})}{\log(\bar{y})}. \quad (17)$$

For fading channels, SNR can be taken as $\bar{y} \sim q(\bar{V})^{-1}$ [15]. Therefore, (17) can be restated as

$$d = \frac{(N-1) \cdot N(N \bar{y} \ln(10))}{(\bar{y} \ln(10))^{-1}} = (N-1). \quad (18)$$

and (12), $P_c(N, \bar{\gamma})$ can be shown as $1 - p(\bar{\gamma})$

$$P_c(N, \bar{\gamma}) = \sum_{N=1}^{\infty} \frac{\phi^N q(\bar{\gamma})^{\frac{-N(N-1)}{2}}}{N!} \quad (13)$$

$$= \log \left(1 + \sum_{i=1}^{\infty} \frac{\phi^i q(\bar{\gamma})^{-i(i-1)/2}}{i!} \right)$$

Since $q(\bar{\gamma}) \in [0, 1]$ and $\phi = 0$, $P_{co}(N, \bar{\gamma}) = 1 - P_c(N, \bar{\gamma})$, the consensus outage probability can be obtained by expanding the logarithm into a power series and grouping coefficients of ϕ^N [10], and the erasure probability can be replaced by the link outage probability in (1), we obtain (10).

This corollary shows that the probability of a graph being strongly connected is related to the random graph equations and it is found that the diversity is equal to the rank of $L(G)$.

V. NUMERICAL RESULTS

In this work, to show that random graphs can be used in the consensus analysis of WMASs, the connection between

random graphs and random behavior of wireless channels is examined and an extensive Monte Carlo simulation is performed for WMASs with varying numbers of agents. The edge connections are generated randomly. The data rate, R , was chosen as 1. The step size, ϵ , was chosen as 0.01 and kept constant during the simulation. We terminate the process

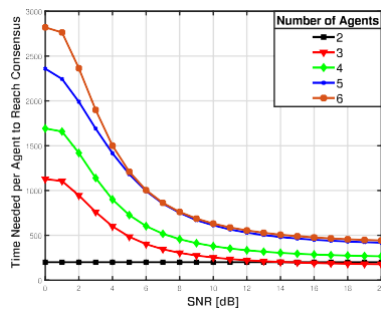


Fig. 3. Time needed per agent to reach consensus for different SNR levels.

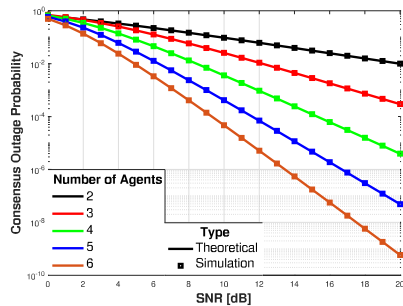


Fig. 4. The consensus outage probability for different SNR levels.

when the error (the difference in agents' values between the two iterations) is under the allowed error threshold, 0.001, or when the maximum number of iterations, 2000, is reached. Theoretical equations for the probability of not having a connected graph in random graphs are derived from (10) with respect to the erasure probability of Rayleigh fading channels as stated in (3). Note that the provided analysis is applicable to different fading distributions, by simply changing $q(\bar{\gamma})$.

Fig. 2 shows the total number of iterations for a different number of agents with respect to different SNR levels. First of all, when there are 2 agents in a WMAS, since there is only one communication link in the system, the number of iterations does not change with SNR. However, in other cases, the number of iterations needed to reach consensus decreases with an increase in the SNR level. For higher SNR levels (more than 10 dB), when the SNR level is kept constant and the number of agents is increased, fewer iterations are needed for a WMAS to reach consensus. For sufficiently high SNR, the number of iterations decreases with an increase in the number of agents in a WMAS. As the number of agents increases, the total number of iterations needed to reach consensus decreases. This outcome is consistent with Corollary 1.

If the total number of agents increases, the time needed to reach consensus will increase due to transmission and iteration computation time. To provide fairness, we also plot the time needed per agent to reach consensus versus SNR in Fig. 3. One key observation we have reached is that, although it sounds counter-intuitive, if the number of agents in a WMAS increases, the graph connectivity increases and this increases the consensus speed for high SNR levels. However, it is also seen that as the SNR level decreases, the time needed to reach consensus will dramatically increase especially for more complex systems. Therefore the SNR level is very crucial for the applicability of complex systems.

In Fig. 4 the consensus outage probability ($P_{Co}(N, \bar{\gamma})$) versus SNR for different number of agents is shown. Both

theoretical and simulation results are plotted. When the SNR value increases, the consensus outage probability decreases for every number of agents. Having more agents in a WMAS with a high SNR environment has the highest probability of reaching consensus. Simulation results match the theoretical results, proving the connection between random graphs and the random behaviors of wireless channels in the consensus analysis of WMASs. The diversity orders are also visible from the slopes of the outage probability curves.

VI. CONCLUSION

The goal of this work is to bridge the literature gap that exists between the consensus problem in WMASs and random graphs. We derive the precise formula for the WMASs consensus probability and validate it using simulation outcomes. Our approach, which considers symmetric channels with the same average SNR values across agents, is based on the literature on random graph theory and models the connection status between agents depending on the channel gains. Furthermore, we establish a direct correlation between diversity and agent count by deriving the corresponding diversity order. Moreover, we found that because of the consensus speed, the SNR level is essential for the applicability of highly populated WMASs. Our goal is to study dynamic networks with time-varying topologies in the future.

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